Answer the following questions about the definition of the definite integral as presented in lecture. **SCORE: _____ / 10 PTS**
(Your answers may refer to the fact that the definite integral equals the area under a curve which is above the $x-$axis.)

[a] What does the formula inside the summation ($f(x_i^*)\,\Delta x$) represent ?

AREA OF EACH RECTANGLE USED TO APPROXIMATE AREA

[b] What is the difference between using $f(x_i^*)$ and $f(a+i\Delta x)$ in the definition ?

x_i^* MEANS ANY POINT IN EACH SUBINTERVAL

$a+i\Delta x$ MEANS ONLY ENDPOINTS

Evaluate the following integrals. **SCORE: _____ / 50 PTS**

[a] $\int_{-3}^{3}(8\sqrt{9-t^2}-\frac{2t^3}{4+3t^2})\,dt$

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$= 8\int_{-3}^{3}\sqrt{9-t^2}\,dt - \int_{-3}^{3}\frac{2t^3}{4+3t^2}\,dt$ 3

$= 8 *$ AREA OF SEMI-CIRCLE OF RADIUS 3

$= 8 * \frac{1}{2}(9\pi)$ 3

$= 36\pi$ 2

$\frac{2(-t)^3}{4+3(-t)^2} = -\frac{2t^3}{4+3t^2}$ 3

2ND INTEGRAND IS ODD, CONTINUOUS 2

INTEGRAL = 0 2

[b] $\int\frac{1-x}{\sqrt{1-2x}}\,dx$

SEE ALSO ALTERNATE SOLUTION

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3 $u = 1-2x \rightarrow x = \frac{1-u}{2}$

$\frac{du}{dx} = -2 \rightarrow dx = -\frac{1}{2}du$

$\frac{1-x}{\sqrt{1-2x}}dx = \frac{1-x}{\sqrt{1-2x}} \cdot -\frac{1}{2}du$

$= \frac{1-\frac{1-u}{2}}{\sqrt{u}} \cdot -\frac{1}{2}du$ 3

$= -\frac{1+u}{4\sqrt{u}}\,du$ 4

$\int(-\frac{1}{4}u^{-\frac{1}{2}} - \frac{1}{4}u^{\frac{1}{2}})\,du$ 3

$= -\frac{1}{2}u^{\frac{1}{2}} - \frac{1}{6}u^{\frac{3}{2}} + C$ 3

$= -\frac{1}{2}(1-2x)^{\frac{1}{2}} - \frac{1}{6}(1-2x)^{\frac{3}{2}} + C$ 3 1

[c] $\int_{0}^{\frac{\pi}{3}}\frac{\sin 2x}{1+\cos^2 x}\,dx$

15

3 $u = 1+\cos^2 x$; $x = \frac{\pi}{3} \rightarrow u = \frac{5}{4}$; $x = 0 \rightarrow u = 2$

$\frac{du}{dx} = 2\cos x(-\sin x)$

$\frac{du}{dx} = -2\sin 2x$

$dx = \frac{du}{-2\sin 2x}$

$\frac{\sin 2x}{1+\cos^2 x}dx = \frac{\sin 2x}{1+\cos^2 x}\,\frac{du}{-2\sin 2x} = -\frac{1}{2}\frac{1}{u}du$ 3 3

$\int_{2}^{\frac{5}{4}} -\frac{1}{2}\frac{1}{u}du = -\frac{1}{2}\ln|u|\Big|_{2}^{\frac{5}{4}} = -\frac{1}{2}(\ln\frac{5}{4} - \ln 2) = -\frac{1}{2}\ln\frac{5}{8} = \frac{1}{2}\ln\frac{8}{5}$

3 2 1

ALTERNATE SOLUTION

[b] $\int \frac{1-x}{\sqrt{1-2x}}\,dx$

4 $u = \sqrt{1-2x} \longrightarrow x = \frac{1-u^2}{2}$

$\frac{du}{dx} = \frac{1}{\cancel{2}\sqrt{1-2x}}\cdot(-\cancel{2})$

$dx = -\sqrt{1-2x}\,du$

$\frac{1-x}{\sqrt{1-2x}}\,dx = \frac{1-x}{\cancel{\sqrt{1-2x}}}\cdot -\cancel{\sqrt{1-2x}}\,du$

$= (x-1)\,du$

4

$= \left(\frac{1-u^2}{2} - 1\right)du$

$= -\frac{1+u^2}{2}\,du$ 5

3

$\int -\frac{1+u^2}{2}\,du = -\frac{1}{2}u - \frac{1}{6}u^3 + C$

$= -\frac{1}{2}\sqrt{1-2x} - \frac{1}{6}\left(\sqrt{1-2x}\right)^3 + C$

3 1

If f is continuous and $\int_3^8 f(x)\,dx = -14$, find $\int_2^3 x f(12 - x^2)\,dx$. **SCORE: / 12 PTS**

(4) $u = 12 - x^2 \quad x=3 \to u=3, \quad x=2 \to u=8$

$\frac{du}{dx} = -2x$

$dx = -\frac{1}{2x}du$

$x f(12-x^2)\,dx = \cancel{x} f(12-x^2)\cdot \frac{-1}{2\cancel{x}}du = -\frac{1}{2}f(u)\,du$

(1) (1) $\int_8^3 -\frac{1}{2}f(u)\,du$ (4) $= \frac{1}{2}\int_3^8 f(u)\,du$

$= \frac{1}{2}(-14)$

$= -7$ (2)

Find $\lim_{n\to\infty} \frac{21}{n}\sum_{i=1}^{n} \frac{1}{\sqrt[3]{1+\frac{7i}{n}}}$ by finding the corresponding definite integral, and evaluating that integral. **SCORE: / 15 PTS**

LET $1+\frac{7i}{n} = a + i\Delta x$

$a = 1$ (3), $\Delta x = \frac{7}{n}$ (3) $= \frac{b-a}{n} \to b = 8$ (3)

$\frac{21}{n} = 3\Delta x$

$\lim_{n\to\infty}\sum_{i=1}^{n} \frac{3}{\sqrt[3]{a+i\Delta x}}\Delta x = \int_1^8 \boxed{\frac{3}{\sqrt[3]{x}}}\,dx$ (3) $= \int_1^8 3x^{-\frac{1}{3}}dx = \frac{9}{2}x^{\frac{2}{3}}\Big|_1^8$ (1)

LET $f(x) = \frac{3}{\sqrt[3]{x}}$

$= \frac{9}{2}(4-1) = \frac{27}{2}$ (2)

The table below gives the rate $p(t)$ at which the pressure of a tank of gas increased as the temperature increased (measured in kPa per degree Celsius). When the temperature was 22° Celsius, the pressure was 180 kPa. **SCORE: / 20 PTS**

t	18	20	22	24	26	28	30	32	34	36	38	40	42	44	46	48	50
~~$r(t)$~~ $p(t)$	1.0	1.2	1.4	1.7	2.0	2.3	2.7	3.1	3.5	3.9	4.4	4.9	5.5	6.2	7.0	7.9	9.0

[a] Write an expression (involving an integral) for the pressure when the temperature was 46° Celsius.

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LET $P(t)$ = PRESSURE WHEN TEMPERATURE IS $t^\circ C$, SO $P'(t) = p(t)$

$\int_{22}^{46} P'(t)\,dt = P(46) - P(22) \to P(46) = 180$ (2) $+ \int_{22}^{46} p(t)\,dt$ (3)

[b] Estimate the pressure when the temperature was 46° Celsius, using the answer to part [a], 3 subintervals and the Midpoint Rule.

15

$\Delta t = \frac{46-22}{3} = 8$ (3) SUBINTERVALS = $[22,30], [30,38], [38,46]$

MIDPOINTS = 26, 34, 42

$180 + (p(26) + p(34) + p(42))\Delta t$ (3)

$= 180 + (2.0 + 3.5 + 5.5)\,8$ (3)

$= 180 + 11 * 8 = 180 + 88$ (3) $= 268$ (2) kPa (1)

Find $\frac{d}{dx}(\cosh^{-1} 2x + \operatorname{sech} x^2)$. **SCORE: ____ / 12 PTS**

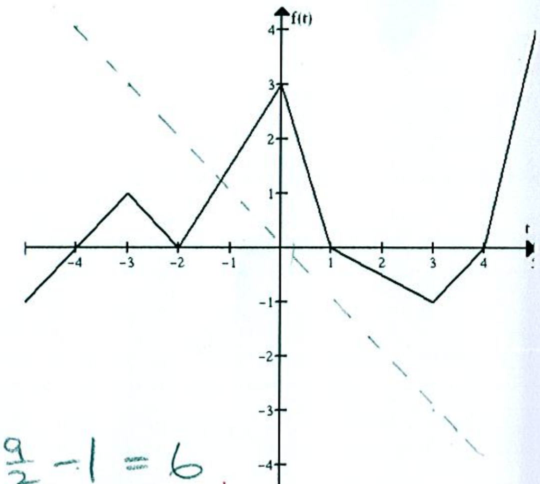
<u>You may use any hyperbolic identities or the derivatives of any hyperbolic functions without proving them.</u>

$$2\tfrac{1}{2}\ \boxed{\frac{1}{\sqrt{(2x)^2-1}}}\underbrace{\cdot 2}_{2}\ \underbrace{-}_{2}\ \underbrace{\operatorname{sech} x^2 \tanh x^2}_{2\frac{1}{2}}\ \underbrace{\cdot 2x}_{2}$$

$$= \frac{2}{\sqrt{4x^2-1}} - 2x \operatorname{sech} x^2 \tanh x^2 \quad +1 \text{ SIMPLIFIED}$$

The graph of $f(t)$ is shown on the right. NOTE: The graph consists of 7 straight line segments. **SCORE: ____ / 31 PTS**

Let $g(x) = \int_{-2}^{x} (t + f(t))\, dt$.



[a] Find $g(3)$.

15

$$\int_{-2}^{3} (t+f(t))\,dt$$

$$= \underbrace{\int_{-2}^{3} t\,dt + \int_{-2}^{3} f(t)\,dt}_{} \quad 3$$

$$2\ \ = \underbrace{\tfrac{1}{2}t^2}_{}\Big|_{-2}^{3} + \int_{-2}^{1} f(t)\,dt + \int_{1}^{3} f(t)\,dt$$

$$= \tfrac{1}{2}(9-4) + \tfrac{1}{2}(3)(3) - \tfrac{1}{2}(2)(1) = \underbrace{\tfrac{5}{2}}_{3} + \underbrace{\tfrac{9}{2}}_{3} \underbrace{-1}_{3} = \underbrace{6}_{1}$$

[b] Find $g'(3)$.

8

$$\underbrace{g'(x) = x + f(x)}_{} \quad 3$$

$$g'(3) = 3 + f(3) = \underbrace{3}_{2} + \underbrace{-1}_{2} = \underbrace{2}_{1}$$

[c] Find $g''(-1)$.

8

$$\underbrace{g''(x) = 1 + f'(x)}_{} \quad 3$$

$$g''(-1) = 1 + f'(-1) = \underbrace{1}_{2} + \underbrace{\tfrac{3}{2}}_{2} = \underbrace{\tfrac{5}{2}}_{1}$$

[d] **<u>OPTIONAL BONUS QUESTIONS:</u>** **BONUS SCORE: _____ / 15 PTS**

[i] Estimate all critical number(s) of g.

$$g'(x) = 0 \longrightarrow \underbrace{x + f(x) = 0}_{1} \longrightarrow \underbrace{f(x) = -x}_{} \quad 2$$

$$\underbrace{x \approx -1\tfrac{1}{3}}_{} \quad 2$$

[ii] Find all inflection points of g.

$$\overset{2}{} \quad \overset{3}{}$$

$$2\ \underbrace{g''(x) \text{ CHANGES SIGNS}}_{} \ @\ x = 0, 1$$

$$1 + f'(x) > 0 \longrightarrow \underbrace{f'(x) > -1}_{} \ 1\tfrac{1}{2} \text{ ON } [-5,-3]\ [-2,0]\ [1,5]$$

$$1 + f'(x) < 0 \longrightarrow \underbrace{f'(x) < -1}_{1\frac{1}{2}} \text{ ON } [0,1]$$